Tutorial 6

March 1, 2017

1. Solve the following general inhomogeneous initial-boundary-value problem for wave equation on half-line:

$$\begin{cases} v_{tt} - c^2 v_{xx} = f(x,t), x > 0, t > 0 \\ v(x,0) = \phi(x), v_t(x,0) = \psi(x), x > 0 \\ v(0,t) = h(t), t > 0 \end{cases}$$

with compatibility conditions $\phi(0) = h(0)$ and $\psi(0) = h'(0)$.

Solution: First, consider the following two problems:

$$\begin{cases} v^1_{tt} - c^2 v^1_{xx} = f(x,t), x > 0, t > 0 \\ v^1(x,0) = \phi(x), v^1_t(x,0) = \psi(x), x > 0 \\ v^1(0,t) = 0, t > 0 \end{cases} \tag{1}$$

and

$$\begin{cases} v^2_{tt} - c^2 v^2_{xx} = 0, x > 0, t > 0 \\ v^2(x,0) = 0, v^2_t(x,0) = 0, x > 0 \\ v^2(0,t) = h(t), t > 0 \end{cases} \tag{2}$$

then $v = v_1 + v^2$ is the solution to original inhomogeneous IBVP.

For problem (1), by reflextion method, the solution formula is given by

$$v_1 = \begin{cases} \frac{1}{2}(\phi(x+ct) + \phi(x-ct)) + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(y)dy + \int_0^t \int_{x-c(t-s)}^{x+c(t-s)} f(y,s)dyds, & x > ct \\ \frac{1}{2}(\phi(x+ct) - \phi(ct-x)) + \frac{1}{2c}\int_{ct-x}^{x+ct} \psi(y)dy + (\int_0^{t-\frac{x}{c}} \int_{c(t-s)-x}^{x+c(t-s)} + \int_{t-\frac{x}{c}}^{t} \int_{x-c(t-s)}^{x+c(t-s)})f(y,s)dyds, & x < ct. \end{cases}$$

For problem (2), the solution has the form of $v_2 = F(x+ct) + G(x-ct)$. The initial conditions imply that for $x > 0$

$$F(x) + G(x) = 0, F'(x) - G'(x) = 0$$

then $F(x) = -G(x) = C$ with constant C for $x > 0$. Let $\tilde{F} = F - C, \tilde{G} = G + C$, then $\tilde{F}(x) = \tilde{G}(x) = 0$ for $x > 0$, and $v_2 = F(x+ct) + G(x-ct) = \tilde{F}(x+ct) + \tilde{G}(x-ct)$. While the boundary condition implies that for $t > 0$

$$\tilde{F}(ct) + \tilde{G}(-ct) = h(t)$$

Notice that $\tilde{F}(x) = 0$ for $x > 0$, thus $\tilde{G}(-ct) = h(t)$, i.e. $\tilde{G}(x) = h(-\frac{x}{c})$ for $x < 0$. Hence the general solution to (2) is

$$v_2 = \begin{cases} 0, & x > ct \\ 0 + \tilde{G}(x-ct) = h(t - \frac{x}{c}), & x < ct \end{cases}$$

Therefore,

$$v = \begin{cases} \frac{1}{2}(\phi(x+ct) + \phi(x-ct)) + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(y)dy + \iint_\Delta f(y,s)dyds, & x > ct \\ \frac{1}{2}(\phi(x+ct) - \phi(ct-x)) + \frac{1}{2c}\int_{ct-x}^{x+ct} \psi(y)dy + \iint_D f(y,s)dyds + h(t - \frac{x}{c}), & x < ct \end{cases}$$

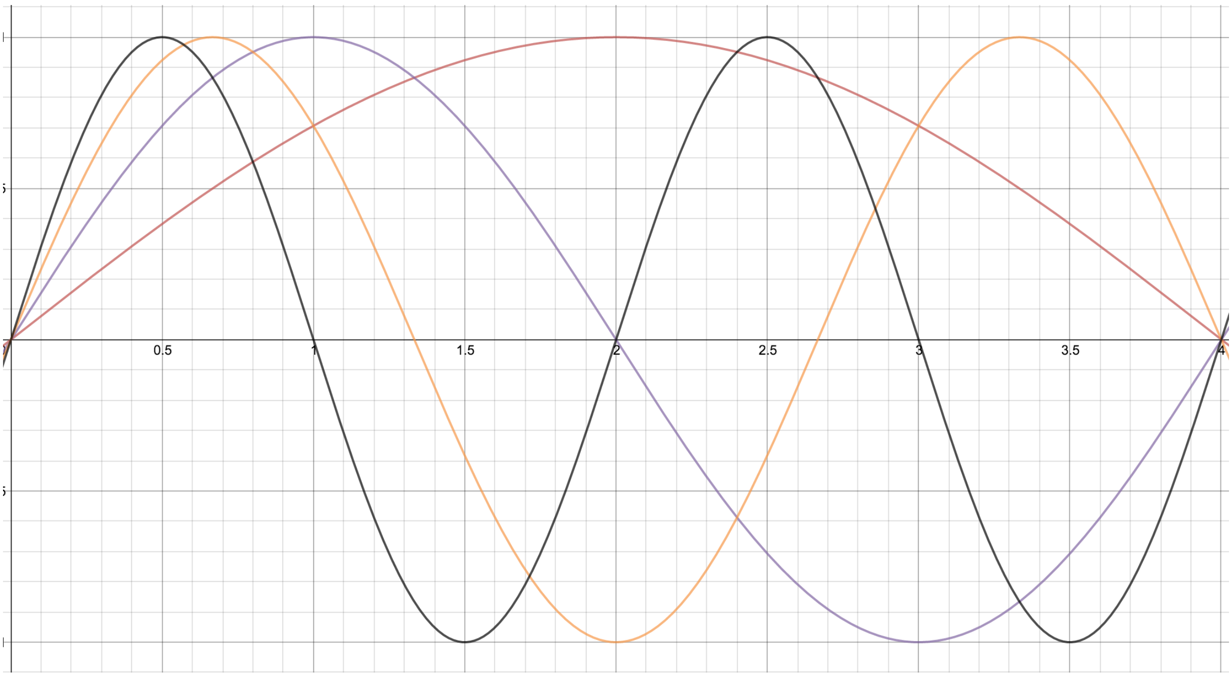

Figure 1: The graphs of eigenfunctions

where Δ and D are characteristic domains as shown in v_1.

2. Discuss the graphs of the eigenfunction $X_n(x) = \sin(\frac{n\pi x}{l})$ for $n = 1, 2, 3, 4$.

   Solution: See figure 1 on page 84 or the above figure: The red line represents $\sin(\frac{\pi x}{l})$, the purple line represents $\sin(\frac{2\pi x}{l})$, the orange line represents $\sin(\frac{3\pi x}{l})$ and the black line represents $\sin(\frac{4\pi x}{l})$.

   Note that the minimal eigenvalue is $(\frac{\pi}{l})^2$ which is called the principal eigenvalue, and its corresponding eigenfunction is $\sin(\frac{\pi x}{l})$ which is always positive when $0 < x < l$.

3. Verify directly that the following eigenvalue problem

$$-X''(x) = \lambda X(x)$$

$$X(0) = X(l) = 0$$

   has no zero or negative eigenvalues.

   Solution: Case 1: If $\lambda = 0$, then $X''(x) = 0$. The general solution is

$$X(x) = ax + b$$

   where a, b are constants. And $X(0) = X(l) = 0$ implies that $a = b = 0$, so that $X(x) = 0$. Therefore 0 is not an eigenvalue.

   Case 2: If $\lambda < 0$, there exists $\gamma > 0$ such that $\lambda = -\gamma^2$. Then $X''(x) - \gamma^2 X(x) = 0$. The general solution is

$$X(x) = Ae^{\gamma x} + Be^{-\gamma x}$$

   where A, B are constants. And $X(0) = X(l) = 0$ implies that $A = B = 0$, so that $X(x) = 0$. Therefore λ can not be negative.

4. Solve the following eigenvalue problem

$$-X''(x) = \lambda X(x)$$

$$X(0) = 0, X'(l) = 0$$

Solution: First, we claim that all eigenvalues are positive.
Multiplying $-X''(x) = \lambda X(x)$ by $\overline{X(x)}$ and integrating w.r.t x from 0 to l give that

$$\lambda \int_0^l |X(x)|^2 dx = -\int_0^l X''(x)\overline{X(x)}dx = -X'(x)\overline{X(x)}\Big|_0^l + \int_0^l |X'(x)|^2 dx = \int_0^l |X'(x)|^2 dx$$

where we have used the boundary conditions. Thus λ must be real and nonnegative. Furthermore, $\lambda = 0$ if and only if $\int_0^l |X'(x)|^2 dx = 0$ which implies that $X'(x) = 0$ and $X(x) = C$. In this case, $X(0) = 0$ tells that $X(x) = C = 0$. Thus the eigenvalue must be positive.

Then, let $\lambda = \beta^2$ with $\beta > 0$. The general solution to $-X''(x) = \lambda X(x)$ is

$$X(x) = A\cos(\beta x) + B\sin(\beta x)$$

Combining with boundary conditions, we have $X(0) = A = 0$ and $X'(l) = \beta B\cos(\beta l) = 0$. Then $\beta l = \frac{\pi}{2} + n\pi$ for $n = 0, 1, 2\cdots$. The eigenvalues are $\lambda_n = (\frac{\pi}{2l} + \frac{n\pi}{l})^2$ and corresponding eigenfunctions are $X_n(x) = \sin((\frac{\pi}{2} + n\pi)x)$, $n = 0, 1, 2\cdots$